Convergence criteria of an infinite continued fraction

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(Received 04 October 1996; revised manuscipt received 22 November 1996)

We discuss the infinite sequence $f(z;\Delta_1,\Delta_2,\Delta_3,\ldots)$ derived by Lee's recurrence relation method [J. Math. Phys. **24**, 2512 (1983)]. We show that the sequence always converges for a nonvanishing real value of z if there are infinitely many Δ_n's which are smaller than some finite value. We also give an argument about the possibility of multivaluedness of the infinite continued fraction. [S1063-651X(97)10503-7]

PACS number(s): 02.70.Rw, 05.70.Ln

I. INTRODUCTION

It is well known that the transport behavior of a dynamical system is characterized by the relaxation mechanism in the system, which can be studied in terms of the relaxation function. The relaxation function is usually expanded in a power series and/or continued fraction (CF), and for real calculation it is advisable that any relevant cutoff stage be found or the power series expansion be modulated for the sake of computational convenience. Most of the expansion is limited to the power series expansion, the reason being that it is quite difficult to find the cutoff stage in the CF. Futhermore, there have been some controversies over the convergence of the expansion [1] and, to our knowledge, the convergence criterion has been introduced in the limited scheme, especially in the CF [2].

The purpose of the present paper is to discuss the convergence of the infinite CF (ICF). Among the many ICF representations introduced so far, here we are interested in the one introduced by Lee [3], since it is easy to deal with mathematically in the Hermitian case, and is quite frequently utilized in the condensed matter physics [4–8]. In Sec. II, the ICF shall be reviewed briefly and the criteria of the convergence shall be introduced for each case of z, the parameter of the Laplace transform of the time-dependent dynamical function. The case for $z=1$ shall be generalized to the case of $z\neq 1$. Finally, the ICF near $z=0$ shall also be discussed. Section III shall be devoted to concluding remarks.

II. CONVERGENCE OF THE ICF

A. Lee's ICF

First we review the recurrence relation introduced by Lee [3] as follows. For a dynamical variable A in a many body system with Hamiltonian H, to which corresponds the Liouville operator L, if $LA=[H,A]\neq 0$, the time evolution in the Heisenberg representation is formally given by

$$A(t)=\exp(iLt)A \tag{1}$$

in the unit system in which $\hbar=1$ where $A(0)=A$. Let there exist a set of orthogonalized basis vectors $\{f_k\}$ spanning the Hilbert space which is realized by the proper inner product (A,B). Here we limit ourselves to the case that A is Hermitian. Discussion for the non-Hermiticity of A can be found in Ref. [9]. It is well known that the basis set $\{f_k\}$ satisfies the recurrence relation

$$f_{k+1}=iLf_k+\Delta_k f_{k-1} \quad (k\geqslant 0), \tag{2}$$

where $\Delta_0=1$ and

$$\Delta_k=\frac{(f_k,f_k)}{(f_{k-1},f_{k-1})} \quad (k\geqslant 1). \tag{3}$$

It is to be noted that here Δ_k corresponds to the Δ_k^2 in the Mori scheme [10] and ν_{k-1}^2 in the Lado-Memory-Parker representation [11], and thus $\Delta_k^{1/2}$ may be called ''the reciprocal time.'' Also note that the characteristic frequencies appearing in Mori's and Lado, Memory, and Parker's representation do not appear here, since we have assumed that A is Hermitian, which makes all the basis vectors Hermitian.

It is also known that $A(t)$ defined in Eq. (1) can be expanded in terms of the basis set $\{f_k\}$ as

$$A(t)=\sum_{k=0}^{\infty} a_k(t)f_k, \tag{4}$$

where $\{a_k(t)\}$ is a set of time-dependent real functions. With the property of $\{f_k\}$ kept in mind, and applying Eq. (2), we obtain the recurrence relation for $\{a_k(t)\}$ as

$$\Delta_{k+1}a_{k+1}(t)=-\frac{da_k(t)}{dt}+a_{k-1}(t) \quad (k\geqslant 0). \tag{5}$$

Note that $a_{-1}(t)=0$ since $f_{-1}=0$. By applying the Laplace transform (L), defined as $\widetilde{g}(z)\equiv Lg(t)=\int_0^\infty \exp(-zt)g(t)dt$ on Eq. (5), we have

$$1=z\widetilde{a}_o(z)+\Delta_1\widetilde{a}_1(z), \tag{6}$$

$$\widetilde{a}_{k-1}(z)=z\widetilde{a}_k(z)+\Delta_{k+1}\widetilde{a}_{k+1}(z) \quad (k\geqslant 1). \tag{7}$$

Equations (6) and (7) can be combined to generate the following continued fraction representation for $\widetilde{a}_o(z)$:

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$$f(z;\Delta_1,\Delta_2,\ldots)=\cfrac{1}{z+\cfrac{\Delta_1}{z+\cfrac{\Delta_2}{z+\ddots}}}. \tag{8}$$

This result is called ''Lee's recurrence relation.''

B. Convergence for $z=1$

Now we will examine the convergence of this scheme. For that purpose we define a sequence $\{f_1,f_2,\ldots\}$ with

$$f_1=\frac{1}{1+\Delta_1},\quad f_2=\cfrac{1}{1+\cfrac{\Delta_1}{1+\Delta_2}},\cdots$$

and, in general,

$$f_j=\cfrac{1}{1+\cfrac{\Delta_1}{1+\cfrac{\Delta_2}{1+\cdots\cfrac{\ddots}{1+\Delta_j}}}}. \tag{9}$$

When $\Delta_n\neq 0$ for all n, this sequence becomes infinite. If $\lim_{n\to\infty}f_n$ exists, then $f(1;\Delta_1,\Delta_2,\ldots)=\lim_{n\to\infty}f_n$, and $f(1;\Delta_1\Delta_2,\ldots)$ is said to be convergent. Conversely, if an infinite sequence is given, we can build $f(1;\Delta_1,\Delta_2,\ldots)$. In other words, $\Delta_1,\Delta_2,\ldots$ can be expressed in terms of $f_1,f_2,\ldots$. From Eq. (9) we obtain

$$\Delta_1=\frac{1}{f_1}-1, \tag{10}$$

$$\Delta_2=\frac{f_2(1-f_1)}{f_1(1-f_2)}-1,\ldots, \tag{11}$$

and, in general,

$$\Delta_n=\frac{(f_n-f_{n-1})(f_{n-3}-f_{n-2})}{(f_n-f_{n-2})(f_{n-1}-f_{n-3})}\quad(n\geqslant 3), \tag{12}$$

where $f_0=1$. The above can be proved easily by the mathematical induction for

$$\Delta_n=\cfrac{\Delta_{n-1}}{\cfrac{\Delta_{n-2}}{\cfrac{\Delta_{n-3}}{\cfrac{\vdots}{\cfrac{\Delta_1}{\cfrac{1}{f_n}-1}-1}-1}-1}-1}-1. \tag{13}$$

Here we introduce a few examples. For $f_n=n+1$, we have $\Delta_1=-1/2$, $\Delta_2=-1/4$, and $\Delta_n(n\geqslant 3)=-1/4$; and thus $f(1;-\frac{1}{2},-\frac{1}{4},-\frac{1}{4},\ldots)=\lim_{n\to\infty}(n+1)$ diverges. For $\Delta_1=\Delta_2=\Delta_3=\ldots=1$, $\{f_1,f_2,\ldots\}=\{\frac{1}{2},\frac{2}{3},\frac{3}{5},\frac{5}{8},\ldots\}$, where $f_n=P_n/P_{n+1}$, P_n being the nth Fibonacci number; $P_1=1,P_2=2,P_3=P_1+P_2,\ldots,P_n=P_{n-2}+P_{n-1}$, known as

$$P_n=\frac{1}{\sqrt{5}}\left[(\sqrt{5}+2)\left(\frac{1+\sqrt{5}}{2}\right)^{n-2}+(\sqrt{5}-2)\left(\frac{1-\sqrt{5}}{2}\right)^{n-2}\right]$$

for $n\geqslant 2$ and

$$\lim_{n\to\infty}\frac{P_n}{P_{n+1}}=\frac{\sqrt{5}-1}{2}.$$

Thus $f(1;1,1,\ldots)=(\sqrt{5}-1)/2$. Note that the convergence cannot be determined only by Δ_n's with large n. For example, $f(1;-1/2,-1/4,-1/4,-1/4,\ldots)$ diverges, while $f(\Delta_1=\Delta_2=\Delta_3=\cdots=-1/4)=2$ converges. Therefore we should investigate the sequence $\{f_1,f_2,\ldots\}$ in terms of given sequence $\{\Delta_1,\Delta_2,\ldots\}$.

However the Δ_n dependence of f_j is not easy to analyze. This is because of the absence of a recurrence relation between f_j and f_{j+1}. In order to investigate the behavior of the infinite sequence further, we define $\delta_j\equiv f_j-f_{j-1}$ and substitute this into Eq. (12). We then obtain

$$\Delta_n=\left(\frac{-\delta_n}{\delta_n+\delta_{n-1}}\right)\left(1+\frac{-\delta_{n-1}}{\delta_{n-1}+\delta_{n-2}}\right), \tag{14}$$

and, letting

$$\gamma_n\equiv\frac{-\delta_n}{\delta_n+\delta_{n-1}}, \tag{15}$$

we obtain

$$\Delta_n=\gamma_n(1+\gamma_{n-1}). \tag{16}$$

For the sake of later convenience we let $\delta_0=1$ (this setting corresponds to $f_{-1}=0$), and from Eq. (15) we obtain $\gamma_1=-\delta_1/(\delta_1+\delta_0)=(1-f_1)/f_1=\Delta_1$. Now we can solve Eq. (16), and express γ_n in terms of Δ_n:

$$\gamma_n=\cfrac{\Delta_n}{1+\cfrac{\Delta_{n-1}}{1+\cdots\cfrac{\Delta_2}{1+\Delta_1}}}. \tag{17}$$

Note that the order of Δ's in Eq. (17) differs from that in Eq. (9). This difference makes the recurrence relation (16) possible.

There is a way to obtain γ_n by a different method. Put $\gamma_n=D_n/N_n$, and substitute this into Eq. (16). We obtain

$$\frac{D_n}{N_n}=\frac{\Delta_nN_{n-1}}{N_{n-1}+D_{n-1}}, \tag{18}$$

or, in matrix form,

$$\begin{pmatrix}N_n\\D_n\end{pmatrix}=\begin{pmatrix}1&1\\\Delta_n&0\end{pmatrix}\begin{pmatrix}N_{n-1}\\D_{n-1}\end{pmatrix}. \tag{19}$$

Iterating Eq. (19), we obtain

$$\begin{pmatrix} N_n \\ D_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \Delta_n & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \Delta_{n-1} & 0 \end{pmatrix} \cdots \begin{pmatrix} 1 & 1 \\ \Delta_2 & 0 \end{pmatrix} \begin{pmatrix} N_1 \\ D_1 \end{pmatrix} \tag{20}$$

where N_1 and D_1 are fixed by $\Delta_1 = D_1/N_1$.

Now we consider δ_n. From Eq. (15) we obtain

$$\delta_n = \frac{-\gamma_n}{1+\gamma_n}\delta_{n-1}. \tag{21}$$

Therefore δ_n can be expressed as a product of γ's,

$$\delta_n = \left(\frac{-\gamma_n}{1+\gamma_n}\right)\left(\frac{-\gamma_{n-1}}{1+\gamma_{n-1}}\right)\cdots\left(\frac{-\gamma_1}{1+\gamma_1}\right). \tag{22}$$

Finally, f_j is given by

$$f_j = \sum_{i=0}^{j} \delta_i. \tag{23}$$

Note that $\delta_0 = f_0 = 1$ in Eq. (23). Therefore the condition $\lim_{n\to\infty}\delta_n = 0$ is a necessary condition for $\lim_{j\to\infty} f_j$ to be finite. In some special cases this condition becomes a necessary and sufficient condition. We now state this in theorem 1.

Theorem 1: If $\Delta_n's$ are real and $\Delta_n > 0$ for all n, the condition $\lim_{n\to\infty}\delta_n = 0$ is a necessary and sufficient condition for $\lim_{j\to\infty} f_j$ to be finite.

Proof: If $\Delta_n's$ are all positive, $\gamma_n's$ are also positive from Eq. (17). Therefore $\delta_1, \delta_2, \delta_3, \ldots$ form an alternating sequence. Furthermore, from Eq. (21) we see $|\delta_n| < |\delta_{n-1}|$ and $|\delta_1|,|\delta_2|,|\delta_3|, \ldots$ form a monotonic decreasing sequence. Consequently $\lim_{j\to\infty} f_j = \Sigma_{n=0}{}^{\infty}\delta_n$ converges, if and only if $\lim_{n\to\infty}\delta_n = 0$ due to the Leibnitz criterion [12].

At this point it may seem that for the cases when $\Delta_n's$ are all positive $\lim_{j\to\infty} f_j$ always converges. This is not true because the condition $\lim_{n\to\infty}\delta_n = 0$ may not be satisfied. Let us give an example where $\lim_{n\to\infty}\delta_n = 0$ is not satisfied. If $\Delta_n's$ are given as

$$\Delta_1 = 3, \quad \Delta_2 = 8, \quad \Delta_3 = 33$$

and

$$\Delta_{2m} = (4m^2 - 4m)(4m^2 - 1)$$

$$\Delta_{2m+1} = (4m^2 + 6m + 1)(4m^2 - 2m - 1), \quad m \geq 2,$$

the corresponding infinite sequence is found to be

$$f_1 = \tfrac{1}{4}, \quad f_2 = \tfrac{3}{4}, \quad f_3 = \tfrac{7}{24}, \quad f_4 = \tfrac{17}{24}, \quad f_5 = \tfrac{11}{36}, \quad f_6 = \tfrac{25}{36}, \ldots,$$

which can be summarized as $f_{2n} = \frac{2}{3} + (1/12n)$ and $f_{2n-1} = \frac{1}{3} - (1/12n)$ for $n \geq 1$. Thus $\lim_{n\to\infty}|\delta_n| = \frac{1}{3}$, and $\lim_{j\to\infty} f_j$ does not exist. However, if some restrictions are imposed on $\Delta_n's$, we may satisfy $\lim_{n\to\infty}\delta_n = 0$. We state this in the next theorem.

Theorem 2: If $\Delta_n's$ are real and positive for all n, and if there are infinite number of $\Delta_k's$ smaller than some finite value, then $\lim_{j\to\infty} f_j$ exists.

Proof: Consider δ_n and assume there are infinite number of $\Delta_k's$ smaller than a finite number, say M. More precisely $\Delta_k < M$ for $k \in I$ (the set of integers) and the number of elements in I is infinite. From Eq. (22),

$$\delta_n = \left(\frac{\gamma_n}{1+\gamma_n}\right)\left(\frac{\gamma_{n-1}}{1+\gamma_{n-1}}\right)\cdots\left(\frac{\gamma_1}{1+\gamma_1}\right) \tag{24}$$

and

$$\delta_n \leq \left(\frac{M}{1+M}\right)^p, \tag{25}$$

where p is the sum of those elements in $I \cap \{1,2,3, \ldots, n\}$. It is obvious that $\lim_{n\to\infty} p = +\infty$ and $M/(1+M) < 1$. Therefore $\lim_{n\to\infty}\delta_n = 0$, and using the previous theorem we obtain the conclusion that $\lim_{j\to\infty} f_j$ exists. This theorem can be proved in a different and easier method. From Eq. (9), when $\Delta_n's$ are real and positive for all n, we see that $f_1 < f_2$ and $f_2 > f_3$ but $f_3 > f_1$. In fact we obtain $f_{2m-1} < f_{2m+1} < f_{2m}$ and $f_{2m-2} > f_{2m} > f_{2m-1}$. Thus $\{f_1, f_3, f_5, \ldots\}$ is a monotonic increasing sequence, while $\{f_2, f_4, f_6, \ldots\}$ is a monotonic decreasing sequence. Furthermore $f_{2m-1} < f_{2n}$ for all natural numbers n and m. From the fact that a monotonic increasing sequence with a upper bound or a monotonic decreasing sequence with a lower bound always converges, we can let

$$\lim_{n\to\infty}\begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \tag{26}$$

where $\alpha_n = f_{2n}$ and $\beta_n = f_{2n-1}$. We also have $\alpha \geq \beta$. From Eq. (12) we have

$$\Delta_{2m} = \frac{(\alpha_m - \beta_m)(\beta_{m-1} - \alpha_{m-1})}{(\alpha_m - \alpha_{m-1})(\beta_m - \beta_{m-1})} \tag{27}$$

and

$$\Delta_{2m+1} = \frac{(\alpha_{m-1} - \beta_m)(\beta_{m+1} - \alpha_m)}{(\alpha_m - \alpha_{m-1})(\beta_{m+1} - \beta_m)}. \tag{28}$$

Now suppose

$$\alpha \neq \beta. \tag{29}$$

Then from Eqs. (27) and (28) we obtain $\lim_{m\to\infty}\Delta_{2m} = \infty$ and $\lim_{m\to\infty}\Delta_{2m+1} = \infty$. Thus $\lim_{n\to\infty}\Delta_n = \infty$, and for our given number M there exists a natural number N such that

$$\Delta_n > M \tag{30}$$

for $n \geq N$. However, this contradicts the given condition that there are infinite number of $\Delta_n's$ which satisfy Eq. (30). Therefore, our supposition (29) was wrong, and we obtain $\lim_{j\to\infty} a_j = \alpha = \beta$.

C. Convergence for $z \neq 1$

Now let us see the z dependence of $f(z;\Delta_1,\Delta_2,\Delta_3,\cdots)$ in Eq. (8). Here we define the corresponding sequence $f_j(z)$ as follows. Assuming $z \neq 0$,

$$f_1(z)=\frac{1}{z+\frac{\Delta_1}{z}},\quad f_2(z)=\frac{1}{z+\frac{\Delta_1}{z+\frac{\Delta_2}{z}}},\quad \dots, \tag{31}$$

and, in general,

$$f_j(z)=\frac{1}{z+\frac{\Delta_1}{z+\frac{\Delta_2}{z+\cdots\frac{\ddots}{z+\frac{\Delta_j}{z}}}}}. \tag{32}$$

As before, we define $f(z;\Delta_1,\Delta_2,\Delta_3,\dots)$ through

$$f(z;\Delta_1,\Delta_2,\Delta_3,\dots)=\lim_{j\to\infty}f_j(z). \tag{33}$$

Thus if $\lim_{j\to\infty}f_j(z)$ exists for a given value of z, we say $f(z;\Delta_1,\Delta_2,\Delta_3,\dots)$ is convergent for that value of z. When $z=1$, we go back to the previous case. Before going further, we first observe that

$$f_j(z)=\frac{1}{z}\frac{1}{1+\frac{\Delta_1/z^2}{1+\frac{\Delta_2/z^2}{1+\cdots\frac{\ddots}{1+\Delta_j/z^2}}}}. \tag{34}$$

Therefore we can let

$$f_j(z)=\frac{1}{z}\tilde{f}_j(z), \tag{35}$$

where $\tilde{f}_j(z)$ is equal to f_j in Eq. (9) except that Δ_n is replaced by Δ_n/z^2. From Eq. (34) we see that $f_j(z)$ is an odd function of z. Therefore $f(z;\Delta_1,\Delta_2,\Delta_3,\dots)$ defined through Eq. (33) is also an odd function of z and satisfies

$$f(z;\Delta_1,\Delta_2,\Delta_3,\dots)=\frac{1}{z}f(1;\Delta_1(z),\Delta_2(z),\Delta_3(z),\dots), \tag{36}$$

where

$$\Delta_n(z)=\frac{\Delta_n}{z^2}. \tag{37}$$

Now we restrict ourselves to the case where z is a nonvanishing real number and, as before, $\Delta_n's$ are real and positive for all n. Note that for nonvanishing real z, $\Delta_n(z)>0$. Therefore the convergence of $f(z;\Delta_1,\Delta_2,\Delta_3,\dots)$ is determined by the convergence of $f(1;\Delta_1(z),\Delta_2(z),\Delta_3(z),\dots)$ which we have so far discussed. Thus we have the following theorem.

Theorem 3: If $\Delta_n's$ are real and positive for all n, and if there are an infinite number of $\Delta_k's$ smaller than some finite value, then $f(z;\Delta_1,\Delta_2,\Delta_3,\cdots)$ is convergent for nonvanishing real z.

Proof: From Eq. (31),

$$f(z;\Delta_1,\Delta_2,\Delta_3,\dots)=\frac{1}{z}f(1;\Delta_1/z^2,\Delta_2/z^2,\Delta_3/z^2,\dots), \tag{38}$$

and if there exists a finite number M which is an upper bound of infinitely many $\Delta_n's$, then M/z^2 is an upper bound of infinitely many $(\Delta_n/z^2)'s$. Thus, from theorem 2, $f(1;\Delta_1/z^2,\Delta_2/z^2,\Delta_3/z^2,\dots)$ is convergent, and consequently $f(z;\Delta_1,\Delta_2,\Delta_3,\dots)$ is also convergent. This completes the proof.

There are several cases which fall into this category. For example, the Laplace transformed time correlation function of the momentum of the impurity particle in an extended Rubin's model [13] is found to be

$$\Xi_{00}(z)=z+\frac{1}{\frac{\lambda^2\mu^2\omega_L^2/2}{z+\frac{\mu^2\omega_L^2/4}{z+\frac{\omega_L^2/4}{z+\frac{\omega_L^2/4}{z+\ddots}}}}}. \tag{39}$$

where λ,μ,ω_L are some characteristic parameters. Here all $\Delta_n's$ are bounded from above, and $\Xi_{00}(z)$ converges for all real z except a possible discontinuity at $z=0$. Another example with a similar structure is found in the Hubbard model [14,15] where the electron correlation function turns out to be

$$\widehat{\Phi}_0(z)=\frac{1}{z+\frac{B^2/4}{z+\frac{B^2/4}{z+\ddots}}}. \tag{40}$$

Here B is some characteristic parameter of this model.

There are many examples where $\Delta_n=n\alpha$ with some fixed positive number α. Among them are a one-dimensional XY model of quantum spin chains [5,16], Gaussian spectrum [6,15], spin van der Waals model [7,8], and so on. In this case the condition of theorem 3 is not satisfied. Let us check the convergence of $f(1;\Delta_1,\Delta_2,\Delta_3,\dots)$ with $\Delta_n=n\alpha$. First we note that $\gamma_n\leqslant\Delta_n$ from Eq. (17). Thus

$$\gamma_n/(1+\gamma_n)\leqslant\Delta_n/(1+\Delta_n), \tag{41}$$

and this implies

$$|\delta_n|\leqslant\zeta_n, \tag{42}$$

where we defined $\zeta_n=(\Delta_n(1+\Delta_n))((\Delta_{n-1})/(1+\Delta_{n-1}))\cdots(\Delta_1/(1+\Delta_1))$. Let us evaluate $\lim_{n\to\infty}\zeta_n$. In order to do that we use

$$\sum_{n=1}^{m}f_n\leqslant\prod_{n=1}^{m}(1+f_n)\leqslant\exp\left(\sum_{n=1}^{m}f_n\right), \tag{43}$$

which holds for $f_n\geqslant0$. If we let $f_n=1/\Delta_n$, we obtain

$$\lim_{n\to\infty} \zeta_n^{-1} = \prod_{n=1}^{\infty} \left(1 + \frac{1}{\Delta_n}\right), \tag{44}$$

and this satisfies

$$\sum_{n=1}^{\infty} \frac{1}{\Delta_n} \leq \lim_{n\to\infty} \zeta_n^{-1}. \tag{45}$$

The left hand side of Eq. (45) diverges when Δ_n is substituted with $n\alpha$. Thus we obtain $\lim_{n\to\infty}\zeta_n = 0$, and consequently $\lim_{n\to\infty}\delta_n = 0$. Therefore when $\Delta_n = n\alpha$, $f(1;\Delta_1,\Delta_2,\Delta_3,\ldots)$ converges. The convergence of $f(z;\Delta_1,\Delta_2,\Delta_3,\ldots)$ for nonvanishing real z follows automatically because $f(1;\Delta_1(z),\Delta_2(z),\Delta_3(z),\ldots)$ converges. In order to prove the convergence of $f(1;\Delta_1(z),\Delta_2(z),\Delta_3(z),\ldots)$, we just change α into α/z^2.

D. ICF near z=0

Finally let us discuss about the behaviour of $f(z;\Delta_1,\Delta_2,\Delta_3,\ldots)$ around $z=0$. Our procedure to obtain $f(z;\Delta_1,\Delta_2,\Delta_3,\ldots)$ through Eq. (33) does not allow us to let $z=0$. Instead we take the limit $z\to 0$. Because $f(z;\Delta_1,\Delta_2,\Delta_3,\ldots)$ is an odd function of z, we expect

$$\lim_{z\to 0^+} f(z;\Delta_1,\Delta_2,\Delta_3,\ldots) = -\lim_{z\to 0^-} f(z;\Delta_1,\Delta_2,\Delta_3,\ldots). \tag{46}$$

Thus if $\lim_{z\to 0^+} f(z;\Delta_1,\Delta_2,\Delta_3,\ldots)$ is not zero, then $f(z;\Delta_1,\Delta_2,\Delta_3,\ldots)$ is not continuous at $z=1$. Here a natural question arises. Which value of the two, either plus value or minus, should be taken as $f(z=0;\Delta_1,\Delta_2,\Delta_3,\ldots)$? In order to answer this question, let us consider the following equation:

$$g^2 + zg = 1. \tag{47}$$

This equation can be written as

$$g = 1/(z+g), \tag{48}$$

and the formal solution to this equation is

$$g = f(z;1,1,1,\ldots) = \cfrac{1}{z + \cfrac{1}{z + \cfrac{1}{z + \ddots}}}. \tag{49}$$

On the other hand, the solutions of Eq. (49) can be obtained easily. They are

$$g = \frac{-z \pm \sqrt{z^2+4}}{2}. \tag{50}$$

Therefore for each z there are two solutions for g. However, we can see easily that our procedure to calculate the infinite continued fraction in Eq. (49) gives only one solution for $f(z;1,1,1,\ldots)$. It is given by

$$f(z;1,1,1,\ldots) = \frac{-z + \sqrt{z^2+4}}{2},$$

when $z>0$, and

$$f(z;1,1,1,\ldots) = \frac{-z - \sqrt{z^2+4}}{2},$$

when $z<0$. If z is a complex number, the function $\sqrt{z^2+4}$ becomes double valued. To make the function single valued we should introduce two branch points at $z=\pm 2i$ and branch cuts starting from these points. Once branch cuts are fixed, then the function becomes single valued and discontinuous along the branch cuts. Our procedure using an infinite sequence to calculate an infinite continued fraction (49) naturally introduces branch cuts. We do not know the location of branch cuts. However we know for sure that one of the branch cuts passes through the origin. This is because $\lim_{z\to 0^+} f(z;1,1,1,\ldots) = 1$, while $\lim_{z\to 0^-} f(z;1,1,1,\ldots) = -1$. The location of branch cuts may be clarified if we analyze (49) for all complex numbers z. Therefore, from example (47) we conclude that whenever $\lim_{z\to 0^+} f(z;\Delta_1,\Delta_2,\Delta_3,\ldots)$ is not zero, the equation for $f(z;\Delta_1,\Delta_2,\Delta_3,\ldots)$ may have some more solutions other than $f(z;\Delta_1,\Delta_2,\Delta_3,\ldots)$ itself.

As an example, we now consider the Rubin model studied by Lee, Florencio, and Hong [17]. We consider a one-dimensional harmonic oscillator chain with spring constant κ and mass m. In the Rubin model, one of the particles is replaced by an impurity of mass m_0. It was shown that the Laplace transform of $\langle P_0(t)P_0\rangle/\langle P_0^2\rangle$, where $P_0(t)$ is the momentum of the tagged mass m_0 at time t and $P_0 = P_0(0)$ and $\langle\ \rangle$ is the classical ensemble average, can be written as an infinite continued fraction

$$\int_0^\infty \frac{\langle P_0(t)P_0\rangle}{\langle P_0^2\rangle} e^{-zt} dt = \cfrac{1}{z + \cfrac{2\lambda\kappa/4}{z + \cfrac{\kappa/4}{z + \ddots}}} \tag{51}$$

where $\lambda = m/m_0$. For simplicity we consider the case when $\lambda = 1$. The right hand side of Eq. (51) with $\lambda = 1$, which we now define to be $f(z)$, converges for real and positive z due to our theorem 3. It can be shown easily that

$$f(z) \equiv \cfrac{1}{z + \cfrac{2\lambda\kappa/4}{z + \cfrac{\kappa/4}{z + \ddots}}} = \frac{1}{\sqrt{z^2+\mu^2}}, \tag{52}$$

where $\mu^2 = 4\kappa/m$ and z, in this expression, is real and positive. From Eq. (52) we see that $\langle P_0(t)P_0\rangle/\langle P_0^2\rangle = J_0(\mu t)$. From the property of the Laplace transformation the left hand side of Eq. (51) is analytic on the half-plane, where $\mathrm{Re}z>0$. Consequently we can conclude that our $f(z)$ in Eq. (52) can be considered as a function of complex z which is analytic on $\mathrm{Re}z>0$. We also know that $f(z)$ as an infinite continued fraction is an odd function of z. Therefore $f(z)$ should be analytic on $\mathrm{Re}z<0$. In order for $f(z)$ to be analytic on the region where $\mathrm{Re}z\neq 0$, the branch cuts, starting from the two branch points $\pm i\mu$, should run along the imaginary axis. One of the branch cuts should pass through the origin. Therefore we conclude that the two branch cuts, starting from $\pm i\mu$, run along the imaginary axis toward $-i\infty$, so that two cuts cancel each other on the line from $-i\mu$ to $-i\infty$. Note that this choice of the branch cuts certifies $f(z)=-f(z)$ and also $f(z)=1/(\sqrt{z^2+\mu^2})$ when z is real and positive.

E. Concluding remarks

So far we have discussed the criteria for convergence of the ICF introduced by Lee's recurrence relation method. We have shown that the ICF (8) converges for all the real and positive $\Delta_n's$ if there are infinitely many $\Delta_n's$ which are smaller than some finite value. We have given some examples which fall into this category. Using our theorems we have shown that the ICF with $\Delta_n = n\alpha$ where α is positive converges. We also have given an argument about the possibility of multivaluedness of the fraction.

The present criteria can be applied to other ICF's [10,11] as far as the same conditions, including the hermiticity of the dynamical value A, are satisfied. The inner products usually adopted in real problems are not positive definite in general. We claim that the convergence of the expansion in those cases should be checked prior to application.

ACKNOWLEDGMENT

This research has been supported by Korea Ministry of Education (BSRI 96-2405).

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